Error Probabilities in Data System Pulse Regenerator with DC Restoration

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Consider a noisy channel which acts as a high-pass filter on the pulses used for transmitting digital data in binary form. To combat the degradation of information in the channel, the pulses are detected and regenerated at certain points with the aid of a binary pulse regenerator with dc restoration. This device achieves complete restoration in the absence of noise.

In this paper, we give a procedure for evaluating the limiting probabilities (after lengthy operation) of error patterns for a single dc restorer in the presence of independent, additive noise. The procedure is based on the observation that for the particular restorer in question, the effective noise in the restorer is the sum of the present noise and the accumulated noise. The latter may be described by a Markovian process.

I. INTRODUCTION

In this paper, we consider a binary pulse regenerator with dc restoration (Fig. 1). This system has the property that in the absence of noise it functions error-free. To evaluate the performance of the system, one would like to know the probability of occurrence for an arbitrary error pattern when noise is introduced. This is highly desirable especially if the information coding places high penalty on certain error patterns. It will be shown in this paper that the limiting probability of any error burst can be computed by an iterative procedure. The mathematical theory justifying the validity of the procedure is somewhat involved and will be given in a separate paper on Random Walk in Compact Metric Space.[1]

II. THE SYSTEM

Our considerations apply to the data transmission system represented by the block diagram given in Fig. 1. The input sequence d_k of random ± 1 impulses goes through a high-pass filter. The output of this filter is

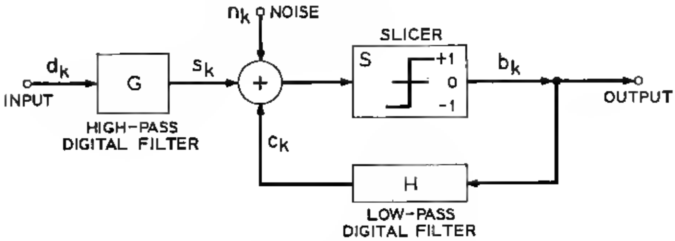

Fig. 1 — Binary pulse regenerator with low-frequency restoration.

sampled synchronously with the impulse train input to yield s_k. It is assumed that the output is contaminated by independent noise n_k. A slicer triggering ± 1 impulses b_k is used for regeneration of pulses. There is a low-pass filter in the feedback loop followed by a sample r which is also synchronized with the input sequence. In the block diagram G and H represent the filters followed by the samplers.

G is a discrete high-pass and H is a discrete low-pass linear filter. S is assumed to be an ideal slicer discriminating between positive and negative voltage levels. Thus, we have the system equations

$$s_k = \sum_{i=0}^{k} g_{k-i} d_i \qquad k = 0,1, \cdots \tag{1}$$

$$c_k = \sum_{i=0}^{k} h_{k-i} b_i \qquad k = 0,1, \cdots \tag{2}$$

and

$$b_k = \operatorname{sgn} \{n_k + c_k + s_k\} \tag{3}$$

where sgn $x = 1$ if $x \geqq 0$ and sgn $x = -1$ if $x < 0$, and $g_i(h_i)$ is the impulse response of the filter $G(H)$ at time i. As to the filters G and H, it is assumed that

(i) $\qquad h_o = 0, \qquad g_0 > 0$

(ii) $\qquad h_i + g_i = 0 \quad \text{if} \quad i \neq 0$

(iii) $\qquad 0 > g_i = rg_{i-1} \quad \text{for} \quad i \geqq 2$

where $0 < r < 1$. The interpretation of $(i$–$iii)$ is as follows: (i) There is unit time delay in the feedback loop. (ii) The filter H is chosen to cancel the tails of the impulse response of G, a condition allowing the error free operation of the system in the absence of noise. (iii) The channel has exponentially decaying impulse response. It is this last

property that will allow us to describe the cumulative error by a Markov process.

The input and noise sequences are regarded as samples from two sequences of completely independent random variables (r.v.). We assume that $d_k = 1$ or $d_k = -1$ with fixed but arbitrary probabilities p and q. The r.v.'s n_k representing the noise have a fixed and, for the time being, arbitrary distribution function (d.f.) $N(x)$. Typically, $N(x)$ is the normal d.f., i.e.,

$$N(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\,(x^2/2\sigma^2 dx). \tag{4}$$

We say that the kth output b_k is in error if $b_k \neq d_k$.

III. EFFECTIVE NOISE

When noise is absent from the system ($n_0 = 0$, $n_1 = 0$, $\cdots$) then $b_k = d_k$ for $k = 0, 1, \cdots$. Clearly, $b_0 = \mathrm{d}_0$. Suppose that $b_i = d_i$ for $i = 0, \cdots, k - 1$. It follows from (1), (2) and (*ii*) that in this case $s_k + c_k = d_k$. Hence, $b_k = d_k$ due to (3). The system is error-free in the absence of noise. To clarify the effect of noise, notice that

$$\begin{aligned} s_k + c_k = &\sum_{\substack{i=0 \\ b_i = d_i}}^{k-1} (h_{k-i}b_i + g_{k-i}d_i) \\ &+ \sum_{\substack{i=0 \\ b_i \neq d_i}}^{k-1} (h_{k-i}b_i + g_{k-i}d_i) + g_o d_k = g_o d_k + x_k, \end{aligned} \tag{5}$$

since $h_i + g_i = 0$, $i \neq 0$, and since $b_i \neq d_i$ implies $b_i = d_i$, it follows that

$$x_k = \sum_{\substack{i=0 \\ b_i = -d_i}}^{k-1} (h_{k-i}b_i + g_{k-i}d_i) = 2 \sum_{\substack{i=0 \\ b_i = -d_i}}^{k-1} g_{k-i}d_i\,. \tag{6}$$

The cumulative effect of errors prior to time k ($d_i \neq b_i$, $i \leqq k - 1$) is expressed by the real number x_k.

The equation of the system, namely (3) now becomes

$$b_k = \operatorname{sgn}\,\{n_k + x_k + g_o d_k\}. \tag{7}$$

Hence, the output b_k will be in error when

$$\begin{aligned} n_k + x_k &< -g_o \quad \text{if} \quad d_k = 1 \\ n_k + x_k &> +g_o \quad \text{if} \quad d_k = -1. \end{aligned} \tag{8}$$

Thus, the effective noise in the system at time k is not n_k but $n_k + x_k$.

Due to the assumptions concerning the independence of all input random variables d_k, the variables n_k and x_k are independent. Let the d.f. of x_k be $F_k(x)$. Also, let $p(k) = \text{prob}\,\{b_k \neq d_k\}$. It follows from (8) that

$$p(k) = p \int N(-g_o - x)dF_k(x) + q \int (1 - N(+g_o - x))dF_k(x). \tag{9}$$

Since p, q, and $N(x)$ are known, the problem reduces to the study of the sequence of r.v.'s $X_0, X_1, \cdots$ with d.f.'s $F_0(x), F_1(x), \cdots$.

It follows from (*iii*) and (6) that

$$\begin{aligned} x_{k+1} &= 2 \sum_{\substack{i=0 \\ b_i \neq d_i}}^{k-1} g_{k+1-i}d_i + 2g_1d_k \\ &= rx_k - ad_k \quad \text{if} \quad d_k \neq b_k . \end{aligned} \tag{10}$$

where $a = -2g_1 > 0$. Or

$$x_{k+1} = rx_k \quad \text{if} \quad d_k = b_k . \tag{11}$$

Thus, there are three possibilities for transitions, each of which takes place with probability depending on the value of x_k. Namely,

$$\begin{aligned} x_{k+1} &= rx_k - a \quad \text{if} \quad d_k = 1 \neq b_k \quad \text{with probability} \quad p_1(x_k), \\ x_{k+1} &= rx_k \quad \text{if} \quad d_k = b_k \quad \text{with probability} \quad p_2(x_k), \\ x_{k+1} &= rx_k + a \quad \text{if} \quad d_k = -1 \neq b_k \quad \text{with probability} \quad p_3(x_k). \end{aligned} \tag{12}$$

The transition probabilities $p_n(x)$, $n = 1,2,3$ are determined from (8):

$$\begin{aligned} p_1(x) &= pN(-g_o - x) \\ p_2(x) &= 1 - p_1(x) - p_3(x) \\ p_3(x) &= q[1 - N(g_o - x)]. \end{aligned} \tag{13}$$

If we assume that the r.v. X_0 is independent of all the input variables, then the sequence of r.v.'s $X_0, X_1, X_2, \cdots$ where x_{k+1} is related to X_k by the transitions characterized by (12) and (13) forms a Markov chain. We shall use this property, however, only to the extent of relations (12) and (13).

Observe the way X_{k+1} is obtained from X_k. Given the value of X_k, the random variable X_{k+1} may have three possible values, the values assumed at X_k by three linear functions defined over the range of X_k.

The choice of the actual transformation used to generate the next value X_{k+1} of the r.v. X_{k+1} is made by performing an independent experiment with three possible outcomes with respective probabilities $p_1(X_k)$, $p_2(X_k)$ and $p_3(X_k)$ which, as indicated, are functions of the value X_k.

IV. ITERATIVE PROCEDURE FOR THE COMPUTATION OF ERROR PROBABILITIES

This type of random walk is studied in Ref. 1. It is shown there that whenever $0 \leqq r < 1$, the sequence of random variables $X_1, X_2, \cdots$ has a limiting distribution $A(x)$ and also that the mean value of any continuous function $f(x)$, with respect to $A(x)$, can be computed by iteration without actually obtaining $A(x)$. This result will now be applied to our prohlem.

Let $Uf(x)$ be the function

$$Uf(x) = p_1(x)f(rx - a) + p_2(x)f(rx) + p_3f(rx + a) \quad (14)$$

where $p_n(x), \quad n = 1,2,3$ are given by (13).

Also, denote by $U^kf(x)$ the kth iterate of the transformation $Uf(x)$, namely,

$$U^kf(x) = U(U^{k-1}f(x)) \qquad k = 1, 2, \cdots. \quad (15)$$

Then, from Ref. 1 we have

$$\lim_k U^kf(x) = \int f(x)dA(x). \quad (16)$$

We then obtain from (9), (13), and (16) that

$$\lim_k p_k = \lim_k U^k(p_1(x) + p_3(x)). \quad (17)$$

For the general case of l consective errors starting with the kth output, we write

$$p(k,l) = \text{Prob}\ (b_k \neq d_k, \cdots, b_{k+l-1} \neq d_{k+l-1}) \quad (18)$$

and for the conditional probability of l consecutive errors given $X_k = x$, we write

$$p(k,l \mid x) = \text{Prob}\ (b_k \neq d_k, \cdots, \quad b_{k+l-1} \neq d_{k+l-1} \mid X_k = x). \quad (19)$$

Clearly,

$$p(k,l) = \int p(k,l \mid x)\ dF_k(x). \quad (20)$$

On the other hand, it follows from (12) and (13) that

$$p(k,l \mid x) = \sum_{\substack{\varepsilon_i \neq 0 \\ i=1,\cdots,l}} p(\varepsilon_1, \cdots, \varepsilon_l \mid x) \tag{21}$$

where

$$p(\varepsilon_1, \cdots, \varepsilon_l \mid x) = p_{2+\varepsilon_1}(x) p_{2+\varepsilon_2}(rx + a\varepsilon_1) \cdots \\ p_{2+\varepsilon_l}(r^{l-1}x + a(r^{l-2}\varepsilon_1 + \cdots + \varepsilon_{l-1})), \tag{22}$$

with $\varepsilon_{i+1} = 1,0,-1$ according as $d_{k+i} = 1 \neq b_{k+i}$, $d_{k+i} = b_{k+i}$, $d_{k+i} = -1 \neq b_{k+i}$. Clearly, $p(k,l \mid x)$ is independent of k. Hence, on account of the theorem used before,

$$q(l) = \lim_k p(k,l) = \lim_k U^k p(k,l \mid x) = \int p(k,l \mid x) dA(x), \tag{23}$$

is the steady state probability of l consecutive errors. Other error patterns may be treated similarly.

V. SUMMARY

Observing that the cumulative error in a certain kind of data transmission system is a Markovian process we have derived an iterative procedure for computing the limiting probability of arbitrary error patterns. Using this method one can obtain numerical estimates of these probabilities by the aid of (23) once a computer program has been written to perform the iteration given in (14). Such a program is not presently available.

A more general treatment of data transmission systems in which error reduction is achieved by quantized feedback may be found in a paper by W. R. Bennett on Synthesis of Active Networks.[2]

VI. ACKNOWLEDGMENT

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REFERENCES

1. Zador, P. L., Random Walk in Compact Metric Space, to be published.
2. Bennett, W. R., Synthesis of Active Networks, Proc. Symposium on Modern Network Synthesis, New York, 1955.